TWO MONOTONIC FUNCTIONS INVOLVING GAMMA FUNCTION AND VOLUME OF UNIT BALL

FENG QI AND BAI-NI GUO

Abstract. In present paper, we prove the monotonicity of two functions involving the gamma function $\Gamma(x)$ and relating to the n-dimensional volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$.

1. Introduction

It is well-known that the classical Euler's gamma function may be defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \tag{1}$$

for $x>0$ and that the n-dimensional volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ is denoted by

$$\Omega_n=\frac{\pi^{n/2}}{\Gamma(1+n/2)},\quad n\in\mathbb{N}. \tag{2}$$

For $x\ge0$, define the function

$$F(x)=\begin{cases}\dfrac{\ln\Gamma(x+1)}{\ln(x^2+1)-\ln(x+1)}, & x\ne 0,1,\\ \gamma, & x=0,\\ 2(1-\gamma), & x=1.\end{cases} \tag{3}$$

Recently, the function $F(x)$ was proved in [7] to be strictly increasing on $[0,1]$. Moreover, as a remark in [7], the function $F(x)$ was also conjectured to be strictly increasing on $(1,\infty)$.

The first aim of this paper is to verify above-mentioned conjecture which can be recited as the following theorem.

Theorem 1. *The function $F(x)$ defined by* (3) *is strictly increasing on* $[0,\infty)$.

The second aim of this paper is to derive the monotonicity of the sequence

$$\Omega_n^{1/[\ln(n^2/4+1)-\ln(n/2+1)]} \tag{4}$$

for $n\in\mathbb{N}$ by establishing the following general conclusion.

2000 *Mathematics Subject Classification.* Primary 33B15; Secondary 26A48.

Key words and phrases. monotonic function, gamma function, volume, unit ball, Descartes' Sign Rule.

The authors were supported in part by the Science Foundation of Tianjin Polytechnic University.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Theorem 2. *The function*

$$G(x)=\left[\frac{\pi^x}{\Gamma(x+1)}\right]^{1/[\ln(x^2+1)-\ln(x+1)]} \tag{5}$$

is strictly decreasing on $(1,\infty)$. *Consequently, the sequence* (4) *is strictly decreasing for* $n\ge 3$.

2. Two lemmas

In order to prove Theorem 1, we need the following lemma which can be found in [13, pp. 9–10, Lemma 2.9], [14, p. 71, Lemma 1] or closely-related references therein.

Lemma 1. *Let* f *and* g *be continuous on* $[a,b]$ *and differentiable on* (a,b) *such that* $g'(x)\ne 0$ *on* (a,b). *If* $\frac{f'(x)}{g'(x)}$ *is increasing (or decreasing) on* (a,b), *then so are the functions* $\frac{f(x)-f(b)}{g(x)-g(b)}$ *and* $\frac{f(x)-f(a)}{g(x)-g(a)}$ *on* (a,b).

We also need the following elementary conclusions.

Lemma 2. *The functions*

$$\begin{aligned}
p_1(x)&=x^3+3x^2-x-1,\\
p_2(x)&=x^3+3x^2-x-1,\\
p_3(x)&=3x^4+8x^3+2x^2-1,\\
p_4(x)&=x^5+3x^4+2x^3+2x^2+x-1,\\
p_5(x)&=x^5+5x^4+6x^3-3x-1,\\
p_6(x)&=120(15-4\ln\pi)x^3+240(20-7\ln\pi)x^2\\
&\quad+48(59-32\ln\pi)x+72(3-4\ln\pi)
\end{aligned}$$

are positive on $(1,\infty)$.

Proof. An easy calculation shows that

$$\begin{aligned}
15-4\ln\pi&=10.421\cdots, & 20-7\ln\pi&=11.986\cdots,\\
59-32\ln\pi&=22.368\cdots, & 3-4\ln\pi&=-1.578\cdots.
\end{aligned}$$

Then Descartes' Sign Rule tells us that the function $p_i(x)$ for $1\le i\le 6$ have just one possible positive root. Since

$$\begin{aligned}
&p_1(0)=-1, && p_2(0)=-1, && p_3(0)=-1, && p_4(0)=-1, && p_5(0)=-1,\\
&p_1(1)=2, && p_2(1)=2, && p_3(1)=12, && p_4(1)=8, && p_5(1)=8,
\end{aligned}$$

and

$$\begin{aligned}
p_6(0)&=-72(4\ln\pi-3)\\
&=-113.68\cdots,\\
p_6(1)&=-[120(4\ln\pi-15)+72(4\ln\pi-3)\\
&\quad+240(7\ln\pi-20)+48(32\ln\pi-59)]\\
&=5087.39\cdots,
\end{aligned}$$

these functions are positive on $[1,\infty)$. $\square$

3. Proof of Theorem 1

The monotonicity of the function $F(x)$ on $[0,1]$ was proved in [7].

For $x\in[1,\infty)$, it is easy to see that

$$\frac{\ln\Gamma(x+1)}{\ln(x^2+1)-\ln(x+1)}=\frac{\ln\Gamma(x+1)-\ln\Gamma(1+1)}{\ln\frac{x^2+1}{x+1}-\ln\frac{1^2+1}{1+1}}=\frac{f(x)-f(1)}{g(x)-g(1)}, \tag{6}$$

where

$$f(x)=\ln\Gamma(x+1)\quad\text{and}\quad g(x)=\ln\frac{x^2+1}{x+1}$$

on $[1,\infty)$. Easy computation and simplification yield

$$\frac{f'(x)}{g'(x)}=\frac{(x+1)\big(x^2+1\big)\psi'(x+1)}{x^2+2x-1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f'(x)}{g'(x)}\right]=\frac{q(x)}{(x^2+2x-1)^2},$$

where

$$\begin{aligned}q(x)&=\big(x^4+4x^3-2x^2-4x-3\big)\psi(x+1)\\&\quad+(x+1)\big(x^2+1\big)\big(x^2+2x-1\big)\psi'(x+1)\end{aligned}$$

and

$$\begin{aligned}q'(x)&=4\big(x^3+3x^2-x-1\big)\psi(x+1)\\&\quad+\big(x^2+2x-1\big)\big[2\big(3x^2+2x+1\big)\psi'(x+1)\\&\quad+(x+1)\big(x^2+1\big)\psi''(x+1)\big].\end{aligned}$$

By virtue of

$$\ln x-\frac1x<\psi(x)<\ln x-\frac1{2x} \tag{7}$$

and

$$\frac{(k-1)!}{x^k}+\frac{k!}{2x^{k+1}}<\big|\psi^{(k)}(x)\big|<\frac{(k-1)!}{x^k}+\frac{k!}{x^{k+1}} \tag{8}$$

for $x>0$ and $n\in\mathbb{N}$, see [3, p. 131], [4, Lemma 3], [9, p. 79], [11, Lemma 3] or related texts in [6, 10], and by using of the positivity of $p_1(x)$ in Lemma 2 and

$$\frac{2t}{2+t}\le\ln(1+t)\le\frac{t(2+t)}{2(1+t)} \tag{9}$$

on $(0,\infty)$, see [5] or [12, p. 245, Remark 1], we obtain

$$\begin{aligned}q'(x)&>4\big(x^3+3x^2-x-1\big)\left[\ln(x+1)-\frac{1}{x+1}\right]\\&\quad+\big(x^2+2x-1\big)\left\{2\big(3x^2+2x+1\big)\left[\frac{1}{x+1}+\frac{1}{2(x+1)^2}\right]\right.\\&\quad\left.-(x+1)\big(x^2+1\big)\left[\frac{1}{(x+1)^2}+\frac{2}{(x+1)^3}\right]\right\}\\&=4\big(x^3+3x^2-x-1\big)\left[\ln(x+1)+\frac{4+x-4x^2+6x^3+16x^4+5x^5}{4(x+1)^2(x^3+3x^2-x-1)}\right]\end{aligned}$$

$$
\begin{aligned}
&\geq 4\bigl(x^3+3x^2-x-1\bigr)\left[\frac{2x}{x+2}+\frac{4+x-4x^2+6x^3+16x^4+5x^5}{4(x+1)^2(x^3+3x^2-x-1)}\right]\\
&=\frac{13x^6+66x^5+86x^4+8x^3-31x^2-2x+8}{(x+1)^2(x+2)}
\end{aligned}
$$

on $[1,\infty)$. Because

$$
\begin{aligned}
13x^6+66x^5+86x^4+8x^3-31x^2-2x+8&=13(x-1)^6+144(x-1)^5\\
&\quad+611(x-1)^4+1272(x-1)^3+1364(x-1)^2+712(x-1)+148>0
\end{aligned}
$$

on $[1,\infty)$, it follows that $q'(x)>0$, and so the function $q(x)$ is increasing, on $[1,\infty)$. Due to

$$
q(1)=8\left(\frac{\pi^2}{6}-1\right)-4(1-\gamma)=3.468\cdots,
$$

the function $q(x)$ is positive on $[1,\infty)$. Therefore,

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f'(x)}{g'(x)}\right]>0
$$

on $[1,\infty)$, which means that the function $\frac{f'(x)}{g'(x)}$ is strictly increasing on $[1,\infty)$. Furthermore, from Lemma 1 and the equation (6), it follows that the function (3) is strictly increasing on $[1,\infty)$. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

Taking the logarithm of the function $G(x)$ and differentiating yield

$$
\ln G(x)=\frac{(\ln\pi)x-\ln\Gamma(x+1)}{\ln(x^2+1)-\ln(x+1)}
$$

and

$$
[\ln G(x)]'=\frac{g(x)}{[\ln(x+1)-\ln(x^2+1)]^2},
$$

where

$$
\begin{aligned}
g(x)&=[\ln\pi-\psi(x+1)]\ln\frac{x^2+1}{x+1}-\frac{x^2+2x-1}{(x+1)(x^2+1)}[(\ln\pi)x-\ln\Gamma(x+1)]\\
&=\frac{x^2+2x-1}{(x+1)(x^2+1)}\biggl\{\frac{(x+1)(x^2+1)[\ln\pi-\psi(x+1)]}{x^2+2x-1}\ln\frac{x^2+1}{x+1}\\
&\quad-(\ln\pi)x+\ln\Gamma(x+1)\biggr\}\\
&\triangleq\frac{x^2+2x-1}{(x+1)(x^2+1)}h(x),
\end{aligned}
$$

with

$$
\begin{aligned}
h'(x)&=\frac{\ln(x+1)-\ln\bigl(x^2+1\bigr)}{(x^2+2x-1)^2}\bigl\{\bigl(x^4+4x^3-2x^2-4x-3\bigr)[\psi(x+1)-\ln\pi]\\
&\quad+(x+1)\bigl(x^2+1\bigr)\bigl(x^2+2x-1\bigr)\psi'(x+1)\bigr\}\\
&\triangleq h_1(x)
\end{aligned}
$$

and

$$
h_1'(x)=4\bigl(x^3+3x^2-x-1\bigr)\psi(x+1)
$$

$$
\begin{aligned}
&+ 2\big(3x^4 + 8x^3 + 2x^2 - 1\big)\psi'(x+1) \\
&+ \big(x^5 + 3x^4 + 2x^3 + 2x^2 + x - 1\big)\psi''(x+1) \\
&- 4\big(x^3 + 3x^2 - x - 1\big)\ln\pi.
\end{aligned}
$$

Utilizing Lemma 2 and employing (7), (8) for $k = 1$ and (9) give

$$
\begin{aligned}
h_1'(x) &> 4\big(x^3 + 3x^2 - x - 1\big)\left[\ln(x+1) - \frac{1}{x+1}\right] \\
&\quad + 2\big(3x^4 + 8x^3 + 2x^2 - 1\big)\left[\frac{1}{x+1} + \frac{1}{2(x+1)^2}\right] \\
&\quad - \big(x^5 + 3x^4 + 2x^3 + 2x^2 + x - 1\big)\left[\frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}\right] \\
&\quad - 4\big(x^3 + 3x^2 - x - 1\big)\ln\pi \\
&= \frac{1}{(x+1)^2}\big[(7 - 4\ln\pi)x^5 + (26 - 20\ln\pi)x^4 - 6(4\ln\pi - 3)x^3 \\
&\quad + (11 + 12\ln\pi)x - 2 + 4\ln(\pi) \\
&\quad + 4\big(x^5 + 5x^4 + 6x^3 - 3x - 1\big)\ln(x+1)\big] \\
&> \frac{1}{(x+1)^2}\bigg[(7 - 4\ln\pi)x^5 + (26 - 20\ln\pi)x^4 - 6(4\ln\pi - 3)x^3 \\
&\quad + (11 + 12\ln\pi)x - 2 + 4\ln(\pi) \\
&\quad + 4\big(x^5 + 5x^4 + 6x^3 - 3x - 1\big)\frac{2x}{x+2}\bigg] \\
&= -\frac{1}{(x+1)^2(x+2)}\big[(4\ln\pi - 15)x^6 + 4(7\ln\pi - 20)x^5 \\
&\quad + 2(32\ln\pi - 59)x^4 + 12(4\ln\pi - 3)x^3 \\
&\quad + (13 - 12\ln\pi)x^2 - 4(3 + 7\ln\pi)x + 4 - 8\ln\pi\big] \\
&\triangleq -\frac{1}{(x+1)^2(x+2)}h_2(x)
\end{aligned}
$$

and

$$
\begin{aligned}
h_2'(x) &= 6(4\ln\pi - 15)x^5 + 20(7\ln\pi - 20)x^4 + 8(32\ln\pi - 59)x^3 \\
&\quad + 36(4\ln\pi - 3)x^2 + 2(13 - 12\ln\pi)x - 4(3 + 7\ln\pi), \\
h_2''(x) &= 30(4\ln\pi - 15)x^4 + 80(7\ln\pi - 20)x^3 \\
&\quad + 24(32\ln\pi - 59)x^2 + 72(4\ln\pi - 3)x + 2(13 - 12\ln\pi), \\
h_2'''(x) &= 120(4\ln\pi - 15)x^3 + 240(7\ln\pi - 20)x^2 \\
&\quad + 48(32\ln\pi - 59)x + 72(4\ln\pi - 3).
\end{aligned}
$$

By Lemma 2, it follows that $h_3''(x)$ is negative on $[1, \infty)$, so $h_2''(x)$ is decreasing and $h_2'(x)$ is concave on $[1, \infty)$. Since

$$
\begin{aligned}
h_2''(1) &= 2(13 - 12\ln\pi) + 30(4\ln\pi - 15) + 72(4\ln\pi - 3) \\
&\quad + 80(7\ln\pi - 20) + 24(32\ln\pi - 59) \\
&= -1696.22\cdots,
\end{aligned}
$$

the derivative $h_2''(x)$ is negative, and thus $h_2(x)$ is concave and $h_2'(x)$ is decreasing, on $[1,\infty)$. From

$$\begin{aligned}h_2'(1)&=2(13-12\ln\pi)+6(4\ln\pi-15)+36(4\ln\pi-3)\\&\quad+20(7\ln\pi-20)-4(3+7\ln\pi)+8(32\ln\pi-59)\\&=-469.89\cdots,\end{aligned}$$

it is immediately deduced that $h_2'(x)$ is negative and the function $h_2(x)$ is decreasing on $[1,\infty)$. Due to

$$\begin{aligned}h_2(1)&=2-16\ln\pi+12(4\ln\pi-3)+4(7\ln\pi-20)\\&\quad-4(3+7\ln\pi)+2(32\ln\pi-59)\\&=-134.10\cdots,\end{aligned}$$

we derive that the function $h_2(x)$ is negative on $(1,\infty)$, so $h_1'(x)>0$ and $h_1(x)$ is increasing on $(1,\infty)$. From

$$h_1(1)=8\biggl(\frac{\pi^2}{6}-1\biggr)-4(1-\gamma)+4\ln\pi=8.04\cdots,$$

it follows that the function $h_1(x)$ is positive on $(1,\infty)$, and thus the derivative $h'(x)$ is negative and $h(x)$ is decreasing on $(1,\infty)$. Since $h(1)=-\ln\pi=-1.14\cdots$, it follows that the function $h(x)$ is negative, that the function $g(x)$ is negative, and that the derivative $[\ln G(x)]'$ is negative on $(1,\infty)$. As a result, the function $G(x)$ is strictly decreasing on $(1,\infty)$.

It is clear that the sequence (4) equals $G\bigl(\frac{n}{2}\bigr)$, so the sequence (4) decreases for $n>2$. The proof of Theorem 2 is complete.

5. Remarks

Remark 1. In [2, Lemma 2.40], it was proved that the sequence $\Omega_n^{1/n}$ decreases strictly to 0 as $n\to\infty$, that the series $\sum_{n=2}^{\infty}\Omega_n^{1/\ln n}$ is convergent, and that

$$\lim_{n\to\infty}\Omega_n^{1/(n\ln n)}=e^{-1/2}. \tag{10}$$

In [1, Corollary 3.1], it was obtained that that the sequence $\Omega_n^{1/(n\ln n)}$ is strictly decreasing for $n\ge2$.

In [8], it was procured that the sequence $\Omega_n^{1/(n\ln n)}$ is strictly logarithmically convex for $n\ge2$.

By L'Hospital rule, we have

$$\lim_{x\to\infty}\ln G(x)=\lim_{x\to\infty}\frac{(\ln\pi)-\psi(x+1)}{(x^2+2x-1)/(x^3+x^2+x+1)}=-\infty,$$

hence,

$$\lim_{x\to\infty}G(x)=\lim_{x\to\infty}\biggl[\frac{\pi^x}{\Gamma(x+1)}\biggr]^{1/\ln\left(\frac{x^2+1}{x+1}\right)}=0$$

and the sequence (4) converges to 0 as $n\to\infty$.

Remark 2. We conjecture that the sequence (4) and the function (5) are both logarithmically convex on $(1,\infty)$.

References

[1] G. D. Anderson and S.-L. Qiu, *A monotoneity property of the gamma function*, Proc. Amer. Math. Soc. **125** (1997), 3355–3362. 6

[2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Special functions of quasiconformal theory*, Expo. Math. **7** (1989), 97–136. 6

[3] B.-N. Guo, R.-J. Chen, and F. Qi, *A class of completely monotonic functions involving the polygamma functions*, J. Math. Anal. Approx. Theory **1** (2006), no. 2, 124–134. 3

[4] B.-N. Guo and F. Qi, *Two new proofs of the complete monotonicity of a function involving the psi function*, Bull. Korean Math. Soc. **47** (2010), no. 1, in press. 3

[5] F. Qi and Ch.-P. Chen, *Monotonicities of two sequences*, Mathematics and Informatics Quarterly **9** (1999), no. 4, 136–139. 3

[6] F. Qi and B.-N. Guo, *A logarithmically completely monotonic function involving the gamma function*, Taiwanese J. Math. **14** (2010), no. 2, in press. 3

[7] F. Qi and B.-N. Guo, *An elegant refinement of a double inequality for the gamma function*, Available online at `http://arxiv.org/abs/1001.1495`. 1, 3

[8] F. Qi and B.-N. Guo, *Monotonicity and logarithmic convexity relating to the volume of the unit ball*, `http://arxiv.org/abs/0902.2509`. 6

[9] F. Qi and B.-N. Guo, *Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic*, Adv. Appl. Math. **44** (2010), no. 1, 71–83; Available online at `http://dx.doi.org/10.1016/j.aam.2009.03.003`. 3

[10] F. Qi and B.-N. Guo, *Sharp inequalities for polygamma functions*, Available online at `http://arxiv.org/abs/0903.1984`. 3

[11] F. Qi, S. Guo and B.-N. Guo, *Complete monotonicity of some functions involving polygamma functions*, J. Comput. Appl. Math. **233** (2010), no. 9, 2149–2160; Available online at `http://dx.doi.org/10.1016/j.cam.2009.09.044`. 3

[12] F. Qi, W. Li, and B.-N. Guo, *Generalizations of a theorem of I. Schur*, Appl. Math. E-Notes **6** (2006), 244–250. 3

[13] F. Qi, D.-W. Niu and B.-N. Guo, *Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl. **2009** (2009), Article ID 271923, 52 pages; Available online at `http://dx.doi.org/10.1155/2009/271923`. 2

[14] F. Qi and A. Sofo, *An alternative and united proof of a double inequality for bounding the arithmetic-geometric mean*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **71** (2009), no. 3, 69–76. 2

(F. Qi) Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China

E-mail address: `qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com`

URL: `http://qifeng618.spaces.live.com`

(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com`